Homework 9 Solution.

1. Consider $\mathbb{Q}^n$ and $\mathbb{R}$ as vectors spaces over $\mathbb{Q}$ (the field of rational numbers). Consider a bilinear form $\mu : \mathbb{Q}^n \times \mathbb{R} \to \mathbb{R}^n$ as follows: $\mu(\mathbf{x}, r) = r\mathbf{x}$ for any $\mathbf{x} \in \mathbb{Q}^n$ and $r \in \mathbb{R}$ (standard scalar multiplication). Let $\beta = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ be the standard ordered basis for $\mathbb{Q}^n$ over the field $\mathbb{Q}$. Suppose γ is a basis for $\mathbb{R}$ over the field $\mathbb{Q}$ (which is infinite dimensional). Prove that $(\mathbb{R}^n, \mu)$ forms a tensor product space. In other words, prove that $\mathbb{Q}^n \otimes \mathbb{R} = \mathbb{R}^n$.

① We show that $\mu(\beta, \gamma)$ is L.I.

Consider

$$a_1 \cdot \mu(e_{m_1}, r_1) + \cdots + a_k \cdot \mu(e_{m_k}, r_k) = 0 \quad ☆$$

where $a_i \in Q$, $e_{m_i} \in \beta$, $r_k \in \gamma$ and $\mu(e_{m_i}, r_i) \neq \mu(e_{m_j}, r_j)$ if $i \neq j$

But e_{m_i} may equals to e_{m_j} for some $i \neq j$.

we group $\{\mu(e_{m_i}, r_i)\}$ based on e_{m_i}

WLOG. ☆ can be written as

$$\left(\sum_{j=1}^{m_1} a_{1,j} \cdot \mu(e_1, r_{1,j}) \right) + \cdots + \left(\sum_{j=1}^{m_n} a_{n,j} \cdot \mu(e_n, r_{n,j}) \right) = 0$$

where $\quad r_{i,j_1} \neq r_{i,j_2}$ if $j_1 \neq j_2$

i.e $\begin{pmatrix} \sum_{j=1}^{m_1} a_{1,j} \cdot r_{1,j} \\ \vdots \\ \sum_{j=1}^{m_n} a_{n,j} \cdot r_{n,j} \end{pmatrix} = \vec{0}$ i.e. $\sum_{j=1}^{m_i} a_{i,j} \cdot r_{i,j} = 0 \quad \forall i = 1, \cdots, n$ ☆

Since γ is a basis, $\{r_{i,1} \cdots r_{i,m_i}\} \subset \gamma$ is L.I.

☆ has a unique solution $a_{i,1} = \cdots = a_{i,m_i} = 0 \quad \forall i = 1 \cdots n.$

Therefore ☆ has a unique zero solution

and $\mu(\beta, \gamma)$ is L.I.

② We show that $\mathbb{R}^n = \mathbb{Q}^n \otimes \mathbb{R}$

- It's obvious that $\mathbb{Q}^n \otimes \mathbb{R} \subset \mathbb{R}^n$

- We show that $\mathbb{R}^n \subset \mathbb{Q}^n \otimes \mathbb{R}$

$\forall\ y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \exists\ q_{i,1} \cdots q_{i,m_i} \in \mathbb{R}$

and $r_{i,1} \cdots r_{i,m_i} \in \gamma$

s.t. $y_i = \sum_{j=1}^{m_i} q_{ij} \cdot r_{ij}$ for $i = 1 \cdots n$

Then $y = \sum_{i=1}^{n} y_i \cdot e_i = \sum_{i=1}^{n} \sum_{j=1}^{m_i} q_{ij} \cdot r_{ij} \cdot e_i$

$= \sum_{i=1}^{n} \sum_{j=1}^{m_i} q_{ij}\ \mu(e_i, r_{ij}) \in \operatorname{span}(\mu(\beta, \gamma))$

Thus $\mathbb{R}^n = \mathbb{Q}^n \otimes \mathbb{R}$

2. The Kronecker product $\mu(A, B)$ of $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{p\times q}$ is defined as:

$$\mu(A,B) = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in M_{mp\times nq}$$

Is the Kronecker product is a tensor product? Justify your answer.

① μ is bi-linear.

$$\mu(cA+A', B) = \begin{bmatrix} (cA_{11}+A'_{11})B & (cA_{12}+A'_{12})B & \cdots & (cA_{1n}+A'_{1n})B \\ \vdots & \vdots & & \vdots \\ (cA_{m1}+A'_{m1})B & (cA_{m2}-A'_{m2})B & \cdots & (cA_{mn}+A'_{mn})B \end{bmatrix}$$

$$= c\begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix} + \begin{bmatrix} A'_{11}B & \cdots & A'_{1n}B \\ \vdots & & \vdots \\ A'_{m1}B & \cdots & A'_{mn}B \end{bmatrix}$$

$$= c\,\mu(A,B) + \mu(A',B)$$

$$\mu(A, cB+B') = \begin{bmatrix} A_{11}(cB+B') & \cdots & A_{1n}(cB+B') \\ \vdots & & \vdots \\ A_{n1}(cB+B') & \cdots & A_{nn}(cB+B') \end{bmatrix}$$

$$= c\begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix} + \begin{bmatrix} A_{11}B' & \cdots & A_{1n}B' \\ \vdots & & \vdots \\ A_{m1}B' & \cdots & A_{mn}B' \end{bmatrix}$$

$$= c\,\mu(A,B) + \mu(A,B')$$

② let $\{E_{ij} : i=1,\cdots,m,\ j=1,\cdots,n\}$ be the standard basis for $M_{m\times n}$

$\{F_{kl} : k=1,\cdots,p,\ l=1,\cdots,q\}$ be the standard basis for $M_{p\times q}$

$$E_{ij} \otimes F_{kl} = \begin{array}{cc} \begin{bmatrix} 0 & \vert & 0 \\ - - & 1 & - - \\ 0 & \vert & 0 \end{bmatrix} & \begin{array}{c} \\ (i-1)p+k \\ \\ \end{array} \\ (j-1)q+l & \end{array} \in M_{mp\times nq}$$

Consider,

$$O_{mp\times nq} = \sum_{i,j,k,l} a_{ij}^{kl} \; E_{ij} \otimes F_{kj} = \begin{bmatrix} a_{11}^{11} & a_{11}^{12} & \cdots & a_{1n}^{1q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}^{p1} & a_{m1}^{p2} & \cdots & a_{mn}^{pq} \end{bmatrix}$$

then $a_{ij}^{kl} = 0 \quad \forall\, i, j, k, l.$

$\therefore \beta = \{ E_{ij} \otimes F_{kj} : 1 \le i \le m,\ 1 \le j \le n,\ 1 \le k \le p,\ 1 \le l \le q \}$ lin. ind.

Besides, $\beta \subset M_{mp\times nq}$ and $|\beta| = mpnq = \dim(M_{mp\times nq})$

$\therefore \beta$ is a basis for $M_{mp\times nq}$.

Thus, kronecker product is a tensor product.

3. Let V and W be two vector spaces over F (They may not be finite-dimensional). Let α be a basis for V, β be a basis for W, and γ be the dual basis of α.

Then $V^* \otimes W = span(\{f \otimes w : f \in \gamma, w \in \beta\})$

Consider the following linear map:

$$\Phi : V^* \otimes W \to \mathcal{L}(V, W)$$

which is defined by

$$\Phi(f \otimes w)(v) = f(v)w \text{ for any } v \in V$$

(a) Show that Φ is one-to-one.

(b) Show that if V is finite-dimensional, then Φ is an isomorphism.

(a) α basis for V. β basis for W. γ dual basis of α.

$\forall g \in N(\Phi) \subset V^* \otimes W$, $\Phi(g) = 0$ the zero transformation

Then $g = \sum_{i=1}^{k} a_i \cdot \gamma_{n_i} \otimes w_{m_i}$ where $a_i \in F$, $\gamma_{n_i} \in \gamma$, $w_{m_i} \in \beta$

$0 = \Phi(g) = \sum_{i=1}^{k} a_i \cdot \Phi(\gamma_{n_i} \otimes w_{m_i})$

i.e. $\sum_{i=1}^{k} a_i \cdot \Phi(\gamma_{n_i} \otimes w_{m_i})(v) = 0 \quad \forall v \in V$

i.e. $\sum_{i=1}^{k} a_i \cdot w_{m_i} \cdot \gamma_{n_i}(v) = 0 \quad \forall v \in V \quad \star$

we group $a_i \cdot w_{m_i} \cdot \gamma_{n_i}(v)$ based on w_{m_i}

WLOG, $\star$ can be written as

$$\sum_{i=1}^{p} \left(\sum_{j=1}^{l_i} a_{ij} \cdot \gamma_{ij}(v) \right) w_i = 0 \quad \forall v \in V$$

since $\{w_1 \cdots w_p\}$ L.I.

we have $\sum_{j=1}^{l_i} a_{ij} \cdot \gamma_{ij}(v) = 0 \quad \forall v \in V$

Since $\{\gamma_{i,1} \cdots \gamma_{i,l_i}\}$ are distinct in γ. thus L.I.

we have $a_{i_1} = \cdots = a_{i,l_i} = 0$ for $i = 1 \cdots p$

Therefore $g = 0$ i.e. Φ is 1-1.

(b) If V is finite-dimensional.

$\forall f \in \mathcal{L}(V, W)$, $\alpha = \{v_1 \cdots v_n\}$ is basis for V

$\gamma = \{\gamma_1 \cdots \gamma_n\}$ is dual basis of α.

define $g = \sum_{i=1}^{n} \gamma_i \otimes f(v_i)$

Then $\Phi(g)(v_j) = \Phi\left(\sum_{i=1}^{n} \gamma_i \otimes f(v_i)\right)(v_j)$

$$= \sum_{i=1}^{n} \gamma_i(v_j) \cdot f(v_i)$$

$$= \sum_{i=1}^{n} \delta_{ij} f(v_i)$$

$$= f(v_j)$$

i.e. $\exists g \in V^* \otimes W$ s.t. $\Phi(g) = f \in \mathcal{L}(V, W)$

Φ is onto, thus an isomorphism.